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approaching completion in the hands of one of us. This memoir will contain an investigation of the systematic errors of the altazimuth observations.

Arkley Observatory,
Barnet, Herts:
1881, *March* 10.

Effect on the Moon's Movement in Latitude, produced by the slow Change of Position of the Plane of the Ecliptic.[*] By Sir G. B. Airy, K.C.B., Astronomer Royal.

It appears that, in consequence of the action of the Planets, the position of the plane of the orbit round the Sun, described by the centre of gravity of the luno-terrestrial system, undergoes a continual change; which, as estimated for a great number of years, may without sensible error be considered as a uniformly progressive change. The plane of the orbit in the year 1900+T years crosses the plane of the orbit in the year 1900 in a line whose longitude (as estimated from the zero of longitude customary in astronomy) is 172° 50′ nearly; and the plane of the orbit is inclined annually by 0″·479, in such a direction that the portion of the plane which is in longitude 262° 50′ is carried towards the north pole of the ecliptic of the year 1900. (Le Verrier, *Annales*, tome ii. p. 105.)

There is no reason to suppose that this small change in the position of the Sun's apparent orbit will produce any discoverable effects on the ecliptic-longitude or radius vector of the Moon; but we may properly examine what is its effect on the Moon's latitude, as referred either to the ecliptic-plane of 1900 or to that of 1900+T years.[†]

Now, referring all forces to the plane of the ecliptic in the year 1900, and putting z_σ for the linear elevation of the Sun at any time above that plane; v being the geocentric longitude of the Moon, and V the longitude of the Sun as viewed from the luno-terrestrial centre of gravity, and other letters having their usual acceptation: it is easily found that—

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[*] On the general question here considered, see *Mécanique Céleste*, Book vii., Chap. i. Also Cayley, "A third Memoir on the Problem of Disturbed Elliptic Motion" (*Mem. R.A.S.* vol. xxxi. [1863], pp. 43–56), and Hansen's *Darlegung der theoretischen Berechnung der in den Mondtafeln angewandten Störungen* (Leipzig, 1862). Laplace's conclusion is that the Motion in regard to the variable ecliptic is the same as it would have been in regard to a fixed ecliptic. Hansen's correction of this result is cited in the footnote further on, p. 271.—Ed.

[†] It appears (p. 491 of the *Darlegung*) that the change produces also a long inequality in the Moon's longitude, the ratio of the coefficients being nearly 8 sin $\frac{1}{2}$J : 1, or numerically about 0·36; but the amount of the irregularity in latitude is not sensibly affected by this circumstance.—Ed.

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Force in z depending on the Sun's elevation above the ecliptic of 1900

$$= +3\frac{\sigma}{A^3} \cdot \left(\frac{A}{R}\right)^3 \cdot \frac{r}{a} \cdot \cos 1 \cdot \cos |\overline{v-V}| \cdot \frac{z_\sigma}{R}.$$

The value of $\frac{\sigma}{A^3}$, using the same constants which I have used in a preceding paper, is $\cdot 0055953$, and $+3\frac{\sigma}{A^3} = +\cdot 016786$.

For the other factors which contain inequalities of the movements of the Earth and Moon, I propose to neglect all those periodic terms which are not strictly and necessarily connected with the present problem, or those whose periodic times are not simply related to those of the principal terms. The only inequalities which are so connected are those depending on $|\overline{v-V}|$ or on $|\overline{nt-Nt}|$, where

$$\frac{N}{n} = \frac{\text{Sun's mean angular motion}}{\text{Moon's mean angular motion}} = \cdot 0748013.$$

For the terms which contain other inequalities, I use only mean values of the coefficients.

Thus we have

$$\left(\frac{A}{R}\right)^3 = 1\cdot 0004215.$$

$$\frac{r}{a} \cdot \cos 1 = \cdot 9995477 - \cdot 0075470 \cdot \cos |\overline{2nt-2Nt}|.$$

$$\left(\frac{A}{R}\right)^3 \frac{r}{a} \cdot \cos 1 = \cdot 9999692 - \cdot 0075501 \cdot \cos |\overline{2nt-2Nt}|.$$

For the two remaining terms, I shall express their value not singly, but in combination. Now, $\frac{z_\sigma}{R}$ is the value (referred to radius) of the Sun's apparent angular elevation above the ecliptic of 1900, or is

$$= +\cdot 479 \times \sin 1'' \times \text{number of years after } 1900 \times \sin |\overline{V-C}|,$$

$$= +\cdot 479 \times \sin 1'' \times \frac{t}{84} \times \sin |\overline{V-C}|,$$ (84 being the number of units of time in a Julian year),

where $C = 172° 50'$.

I shall use the letter B for $+\cdot 479 \times \sin 1'' \times \frac{1}{84}$. Then the Sun's angular elevation above the ecliptic $= Bt . \sin |\overline{V-C}|$; and the product of the two last terms in the expression for "Force in z" is

$$\frac{Bt}{2} \times \left\{ +\sin |\overline{v-C}| - \sin |\overline{v-2V+C}| \right\}.$$

Now, $v = nt + \cdot 01148 . \sin |\overline{2nt-2Nt}|$ (including only those

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small terms of which we have spoken), and V has no term depending on $\overline{nt-\mathrm{N}t}$; and the product of the two terms therefore is

$$+\frac{\mathrm{B}t}{2}\times\left\{\begin{array}{l}+\sin\overline{nt-\mathrm{C}}-\sin\overline{nt-2\mathrm{N}t+\mathrm{C}}\\+\left(\cos\overline{nt-\mathrm{C}}-\cos\overline{nt-2\mathrm{N}t+\mathrm{C}}\right)\times\cdot 01148\,.\sin\overline{2nt-2\mathrm{N}t}\end{array}\right\}$$

$$=\frac{\mathrm{B}t}{2}\times\left\{\begin{array}{l}+\sin\overline{nt-\mathrm{C}}-\sin\overline{nt-2\mathrm{N}t+\mathrm{C}}\\+\cdot 00574\,.\sin\overline{3nt-2\mathrm{N}t-\mathrm{C}}+\cdot 00574\,.\sin\overline{nt-2\mathrm{N}t+\mathrm{C}}\\-\cdot 00574\,.\sin\overline{3nt-4\mathrm{N}t+\mathrm{C}}-\cdot 00574\,.\sin\overline{nt-\mathrm{C}}\end{array}\right\}$$

Multiplying this by

$$+\cdot 016786\times\{+\cdot 99997-\cdot 00755\,.\cos\overline{2nt-2\mathrm{N}t}\}$$

(found above for $3\frac{\sigma}{\mathrm{A}^3}\times\left(\frac{\mathrm{A}}{\mathrm{R}}\right)^3\cdot\frac{r}{a}\cos l$), we have for the force in z,

$$+\cdot 0083930\,.\,\mathrm{B}t\times\left\{\begin{array}{l}+\cdot 99997\,.\sin\overline{nt-\mathrm{C}}-\cdot 99997\,.\sin\overline{nt-2\mathrm{N}t+\mathrm{C}}\\+\cdot 00197\,.\sin\overline{3nt-2\mathrm{N}t-\mathrm{C}}+\cdot 00951\,.\sin\overline{nt-2\mathrm{N}t+\mathrm{C}}\\-\cdot 00197\,.\sin\overline{3nt-4\mathrm{N}t+\mathrm{C}}-\cdot 00951\,.\sin\overline{nt-\mathrm{C}}\end{array}\right\}$$

$$=\mathrm{B}t\times\left\{\begin{array}{l}+\cdot 0083130\,.\sin\overline{nt-\mathrm{C}}-\cdot 0083130\,.\sin\overline{nt-2\mathrm{N}t+\mathrm{C}}\\+\cdot 0000166\,.\sin\overline{3nt-2\mathrm{N}t-\mathrm{C}}-\cdot 0000166\,.\sin\overline{3nt-4\mathrm{N}t+\mathrm{C}}\end{array}\right\}$$

This is the expression for the force which is to be used on the first side of the Equation of the Factorial Table.*

The equation is the following—

$$\text{Force}=\left\{\begin{array}{l}+\;\delta l\times(+1\cdot 007945+\cdot 0428\,.\cos\overline{2nt-2\mathrm{N}t})\\+\frac{d\delta l}{dt}\times(\quad\cdot 000000+\cdot 0278\,.\sin\overline{2nt-2\mathrm{N}t})\\+\frac{d^2\delta l}{dt^2}\times(+\;\cdot 999545-\cdot 0075\,.\cos\overline{2nt-2\mathrm{N}t})\end{array}\right\}$$

A very little trial and consideration will show that the form to be assumed for δl, whose substitution in the last expression will produce such a "force" as was found above, must be the following—

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* The Factorial Table to which reference is made here, is a Table embodying three equations; which are prepared by forming the three equations of motion, each of which is expressed in terms of r, v, and z; and differentiating these equations (including both the geometrical ordinates, and the terrestrial and solar mechanical forces due to these geometrical ordinates) in such a manner as to exhibit the magnitudes of new forces which must be introduced in order to produce ordinates $r+\delta r$, $v+\delta v$, $z+\delta z$. I hope shortly to publish in detail, in a "Numerical Lunar Theory," the steps of formation of these equations. In the present instance, only one of the equations of motion is employed, and it is brought to its simplest state by neglecting all variable quantities not necessarily involved in the problem.

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$$\delta l = \left\{\begin{array}{l} +Pt\,.\sin\left|\overline{nt-C}\right| + Qt\,.\sin\left|\overline{nt-2Nt+C}\right| + Rt\,.\sin\left|\overline{3nt-2Nt-C}\right| + St\,.\sin\left|\overline{3nt-4Nt+C}\right| \\ +\,p\,.\cos\left|\overline{nt-C}\right| +\,q\,.\cos\left|\overline{nt-2Nt+C}\right| +\,r\,.\cos\left|\overline{3nt-2Nt-C}\right| +\,s\,.\cos\left|\overline{3nt-4Nt+C}\right| \end{array}\right\},$$

where, in each line, the third and fourth terms are much inferior in magnitude to the first and second, and their products with the small terms of the Factorial Equations are to be neglected. By differentiating δl we obtain

$$\frac{d\delta l}{dt} = \left\{\begin{array}{rr} +Pt\,.\cos\left|\overline{nt-C}\right| & +{\cdot}8504\,.\,Qt\,.\cos\left|\overline{nt-2Nt+C}\right| \\ +(P-p)\,.\sin\left|\overline{nt-C}\right| & +(Q-{\cdot}8504\,.\,q)\,.\sin\left|\overline{nt-2Nt+C}\right| \\ +2{\cdot}8504\,.\,Rt\,.\cos\left|\overline{3nt-2Nt-C}\right| & +2{\cdot}7008\,.\,St\,.\cos\left|\overline{3nt-4Nt+C}\right| \\ +(R-2{\cdot}8504\,.\,r)\,.\sin\left|\overline{3nt-2Nt-C}\right| & +(S-2{\cdot}7008\,.\,s)\,.\sin\left|\overline{3nt-4Nt+C}\right| \end{array}\right\},$$

$$\frac{d^2\delta l}{dt^2} = \left\{\begin{array}{rr} -Pt\,.\sin\left|\overline{nt-C}\right| & -{\cdot}7231\,.\,Qt\,.\sin\left|\overline{nt-2Nt+C}\right| \\ +(2P-p)\,.\cos\left|\overline{nt-C}\right| & +(1{\cdot}7008\,.\,Q-{\cdot}7231\,.\,q)\cos\left|\overline{nt-2Nt+C}\right| \\ -8{\cdot}1247\,.\,Rt\,.\sin\left|\overline{3nt-2Nt-C}\right| & \\ +(5{\cdot}7008\,.\,R-8{\cdot}1247\,.\,r)\,.\cos\left|\overline{3nt-2Nt-C}\right| & +(5{\cdot}4016\,.\,S-7{\cdot}2944\,.\,s)\,.\cos\left|\overline{3nt-4Nt+C}\right| \end{array}\right\}.$$

It is readily seen that wherever p appears it is accompanied with a multiple of P: and so for q, r, s. Then, when P, Q, R, S, are found from the terms which multiply t, and are substituted in those which do not contain t, but which are easily solved, p, q, r, s, will be directly deduced from them. We may therefore, in the first instance, omit entirely the terms independent of t. We proceed to substitute the terms depending on t in the factorial equation.

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External force, representing the physical assumptions; or First Side of the Factorial Equation:—

$$= \mathrm{B}t \times \{+\cdot 0083130 \,.\, \sin \overline{nt-\mathrm{C}} - \cdot 0083130 \,.\, \sin \overline{nt-2\mathrm{N}t+\mathrm{C}} + \cdot 0000166 \,.\, \sin \overline{3nt-2\mathrm{N}t-\mathrm{C}} - \cdot 0000166 \,.\, \sin \overline{3nt-4\mathrm{N}t+\mathrm{C}}\}.$$

Corresponding terms of the Second Side of the Factorial Equation, including only multiples of t:—

$$= \left\{ \begin{array}{l} +\{+1\cdot 007945 + \cdot 0428 \,.\, \cos \overline{2nt-2\mathrm{N}t}\} \\ \qquad \times t \times \{+\mathrm{P} \,.\, \sin \overline{nt-\mathrm{C}} \quad + \,.\, \mathrm{Q} \,.\, \sin \overline{nt-2\mathrm{N}t+\mathrm{C}} \quad + \mathrm{R} \,.\, \sin \overline{3nt-2\mathrm{N}t-\mathrm{C}} \quad + \mathrm{S} \,.\, \sin \overline{3nt-4\mathrm{N}t+\mathrm{C}}\} \\ +\{+\ \cdot 000000 + \cdot 0278 \,.\, \sin \overline{2nt-2\mathrm{N}t}\} \\ \qquad \times t \times \{+\mathrm{P} \,.\, \cos \overline{nt-\mathrm{C}} + \cdot 8504 \,.\, \mathrm{Q} \,.\, \cos \overline{nt-2\mathrm{N}t+\mathrm{C}} + 2\cdot 8504 \,.\, \mathrm{R} \,.\, \cos \overline{3nt-2\mathrm{N}t-\mathrm{C}} + 2\cdot 7008 \,.\, \mathrm{S} \,.\, \cos \overline{3nt-4\mathrm{N}t+\mathrm{C}}\} \\ +\{+\ \cdot 999545 - \cdot 0075 \,.\, \cos \overline{2nt-2\mathrm{N}t}\} \\ \qquad \times t \times \{-\mathrm{P} \,.\, \sin \overline{nt-\mathrm{C}} - \cdot 7231 \,.\, \mathrm{Q} \,.\, \sin \overline{nt-2\mathrm{N}t+\mathrm{C}} - 8\cdot 1247 \,.\, \mathrm{R} \,.\, \sin \overline{3nt-2\mathrm{N}t-\mathrm{C}} - 7\cdot 2944 \,.\, \mathrm{S} \,.\, \sin \overline{3nt-4\mathrm{N}t+\mathrm{C}}\} \end{array} \right\}.$$

On performing the multiplications in these lines, omitting the products of small terms of the first factors by the coefficients of R and S in the second factors, the Second Side of the Factorial Equation becomes

$$+\left\{\begin{array}{l} +\cdot 008400 \,.\, \mathrm{P} \\ -\cdot 0123 \quad .\, \mathrm{Q} \end{array}\right\} t \,.\, \sin \overline{nt-\mathrm{C}} \qquad + \left\{\begin{array}{l} -\cdot 0114 \,.\, \mathrm{P} \\ +\cdot 2851 \,.\, \mathrm{Q} \end{array}\right\} t \,.\, \sin \overline{nt-2\mathrm{N}t+\mathrm{C}}$$

$$+\left\{\begin{array}{l} +\ \cdot 0392 \cdot \mathrm{P} \\ -7\cdot 1130 \cdot \mathrm{R} \end{array}\right\} t \,.\, \sin \overline{3nt-2\mathrm{N}t-\mathrm{C}} \qquad + \left\{\begin{array}{l} +\ \cdot 0359 \,.\, \mathrm{Q} \\ -6\cdot 2832 \,.\, \mathrm{S} \end{array}\right\} t \,.\, \sin \overline{3nt-4\mathrm{N}t+\mathrm{C}}.$$

This is to be compared with the First Side of the Factorial Equation.

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Solving the four equations formed by comparison of the coefficients of sines of similar arguments,

$$P = +B \times 1{\cdot}003, \qquad Q = +B \times {\cdot}011, \qquad R = +B \times {\cdot}005, \qquad S \text{ is insensible.}$$

These, it will be remembered, are the important part of the result of the investigation as giving those term-coefficients which increase with time.

We now proceed with the less important terms whose coefficients do not increase with time, but which represent perturbations recurring with similar form and magnitude at every recurrence of similar positions of the Sun and Moon.

On the First Side of the Factorial Equation there are no coefficients independent of t. And there is no doubt that, in general, the terms of the result for constant coefficients will be so small that the retention of the principal part of each factor on the second side will be sufficiently accurate. There is one exception to this; that for each argument there are two coefficients, whose difference enters importantly, as divisor, into the equation; and in the special instance of the coefficients of cos $\overline{nt-C}|$, the magnitudes of those coefficients are so nearly equal, and their difference (or divisor of the equation) so small, that in order to produce an available result, a greater number of decimals must be retained in their values. Remembering that the terms multiplying t have been entirely cleared out, the equations now to be treated are contained in these two lines—

$$0 = \left\{ \begin{array}{llll} +\,1{\cdot}007945\,.\,p\,.\cos\overline{nt-C}| & \qquad +\,q)\,.\cos\overline{nt-2Nt+C}| & & \\ +\ {\cdot}999545\,.\,(2P-p)\,.\cos\overline{nt-C}| & +\,(1{\cdot}7008\,.\,Q-{\cdot}7231\,.\,q)\,.\cos\overline{nt-2Nt+C}| & & \\ & \qquad +\,r\,.\cos\overline{3nt-2Nt-C}| & & +\,S\,.\cos\overline{3nt-4Nt+C}| \\ & +\,(5{\cdot}7008\,.\,R-8{\cdot}1247\,.\,r)\,.\cos\overline{3nt-2Nt-C}| & & +\,(5{\cdot}4016\,.\,S-7{\cdot}2944\,.\,s)\,.\cos\overline{3nt-4Nt+C}| \end{array} \right\}.$$

Referring to the late investigations of the values of R and S, it will be seen that r and s are totally unimportant; even q is far too small to be recognised in observation. But p is more important. Its equation is

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$-1{\cdot}999090.P + {\cdot}008400.p = 0$, or $p = -238 \times P$, which is sensibly equal to $-238 \times B$, or to $-238 \times {\cdot}479 \times \sin 1'' \times \frac{1}{84}$; or in seconds of arc, $p = -1''{\cdot}36$.

The physical result of the investigation, as regards the terms whose coefficients depend on time, may be thus presented. The elevation above the fixed ecliptic of 1900, of that part of the moveable ecliptic of $1900 + t$, near which the Moon happens to be at the time t, is $Bt.\sin\overline{|v - C|}$, or

$$Bt \times \left\{ + \sin\overline{|nt - C|} + {\cdot}00574 . \sin\overline{|nt - 2Nt + C|} + {\cdot}00574 \sin\overline{|3nt - 2Nt - C|} \right\}.$$

Thus we have

$$\left.\begin{array}{l}\text{Perturbation of the Moon at the time } t \text{, above the fixed} \\ \quad\text{ecliptic of 1900.}\end{array}\right\} = Bt \times \left\{ + 1{\cdot}003 . \sin\overline{|nt - C|} + {\cdot}011 . \sin\overline{|nt - 2Nt + C|} + {\cdot}005 . \sin\overline{|3nt - 2Nt - C|} \right\},$$

$$\left.\begin{array}{l}\text{Elevation above the fixed ecliptic of 1900, of that part} \\ \quad\text{of the moveable ecliptic of } (1900 + t) \text{ near which} \\ \quad\text{the Moon happens to be at the time } t.\end{array}\right\} = Bt \times \left\{ + \sin\overline{|nt - C|} + {\cdot}006 . \sin\overline{|nt - 2Nt + C|} + {\cdot}005 . \sin\overline{|3nt - 2nt - C|} \right\},$$

$$\left.\begin{array}{l}\text{Excess of the first line or Perturbation of the Moon at} \\ \quad\text{the time } t \text{ above the moveable ecliptic at the time } t.\end{array}\right\} = Bt \times \left\{ + {\cdot}003 . \sin\overline{|nt - C|} + {\cdot}005 . \sin\overline{|nt - 2Nt + C|} \right\}.$$

The last figures I believe to be purely accidental, arising from the use of decimals to a point almost beyond what they can bear. Rejecting them, the general result is—

(A.) The Moon's movements, in so far as they might be supposed to depend on coefficients increasing with time, are the same with reference to a disturbed ecliptic as they would be with reference to an undisturbed ecliptic. (The immediate relative action of planets upon the Earth and Moon is not considered.)

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But there is also a term $-1''\cdot36 \cos \left|\overline{nt-C}\right|$ in the Moon's latitude, with invariable coefficient, which may be expressed in the form

$$-\overset{''}{1}\cdot36 \,.\, \sin \left|\overline{\text{Moon's longitude} - 82^\circ\ 50'}\right|$$

or

$$+\overset{''}{1}\cdot36 \,.\, \sin \left|\overline{\text{Moon's longitude} - 262^\circ\ 50'}\right|$$

Comparing this with the assumptions at the beginning, we see that—

(B.) The movements of the Moon, as expressed by the ordinary theories, are to be referred, not to the moveable ecliptic, but to a plane which is inclined to the moveable ecliptic in an angle of $1''\cdot36$; the axis of inclination being at right angles to the axes of rotation of the moveable ecliptic on the fixed ecliptic, and the greatest depression taking place at longitude $172''\ 50'$.*

We may express the term

$$-\overset{''}{1}\cdot36 \,.\, \cos \left|\overline{\text{Moon's longitude} - 172^\circ\ 50'}\right|$$

under the equivalent form

$$+\overset{''}{1}\cdot35 \,.\, \cos \text{Moon's longitude} - \overset{''}{0}\cdot17 \,.\, \sin \text{Moon's longitude}.$$

Now, on referring to the discussion of the Greenwich lunar observations in the *Memoirs* of this Society, volume xxix. page 23, it will be seen that the observations give an equation in latitude

$$+\overset{''}{1}\cdot93 \,.\, \cos \text{Moon's longitude} - \overset{''}{0}\cdot59 \,.\, \sin \text{Moon's longitude},$$

(an equation which was the source of anxious consideration to Professor Hansen and myself). I do not doubt that this equation is completely explained by the result of the preceding theory. The law of connection of the two terms is well in accord; the magnitude of the coefficients is not perfectly in agreement, but not so discordant as at first appears (since the coefficients from the observations are the results of observed coefficients, every one of which is far inferior in merit to the exhibited result). It may be well to examine severely the factors in the investigation cited above.

1881, *February* 1.

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* In the *Darlegung*, p. 118, Hansen arrives at the conclusion that, disregarding periodic inequalities, the Moon moves in an orbit having a constant inclination, "nicht auf der gleichzeitig stattfindenden Ecliptik, sondern auf der welche nahe drei Jahre vorher Statt fand, und deren Knoten mit der gleichzeitigen Ecliptik überdies um einen Quadranten zurück gewichen sind." See also p. 494 of the *Darlegung*, where reference is made to the equation in latitude obtained from the Greenwich Lunar Observations.—Ed.